

A CRITIQUE OF THE SCHOOLS OF THOUGHT IN PHILOSOPHY OF MATHEMATICS

Anthony Raphael Etuk, PhD.

University of Uyo

Akwa Ibom State.

E-mail: etuky4real@gmail.com

ORCID id: <https://orcid.org/0000-0002-7708-9061> .

Abstract

The discovery of paradoxes concerning certain mathematical assumptions in the wake of the twentieth century, posed a serious challenge to the objectivity and certainty of mathematical knowledge. This not only ushered in scepticism in the mathematical discipline, but also gave rise to the profound activity on meta-mathematics geared towards greater research on the foundations of mathematics. Several schools of thought in the Philosophy of mathematics emerging in the sequel, have sought to clarify the foundations and the nature of mathematical knowledge. This paper explores the positions of the five major such schools of thought in the philosophy of Mathematics, namely: Logicism, Intuitionism, Empiricism, Formalism, Psychologism. The paper argues that notwithstanding their relevant contributions in clarifying the foundations of mathematics, none of these schools meets the requirement of rational success in adequately addressing the challenge to the objectivity and certainty of mathematical knowledge, since each is nettled with some identified defects. The paper concludes with the conviction that greater research in this aspect of philosophy of mathematics will yield the required desideratum. The expository and analytic methods of research are adopted in the paper.

Keyword: Mathematics, Logicism, Intuitionism, Empiricism, Formalism, Psychologism.

Introduction

For a larger part of the 20th century till date, the philosophy of

mathematics has been commonly represented as a debate among various schools of thought concerning the nature of mathematics and the source of mathematical knowledge as well as attempts to resolved perceived paradoxes involving mathematical assumptions. It is a profound paradox that on the one hand, mathematical truths seem to have compelling inevitability, but on the other hand, the source of their “truthfulness” remains quite elusive. Emerging mostly in response to the increasingly widespread worry that mathematics as it stood, did not live up to the standard of certainty and rigor that has been accorded it, these competing schools of thought adopt different approaches to these issues in explaining the nature and source of mathematical knowledge.

Broadly distinguished by their pictures of mathematical epistemology and ontology, these schools of thought put forth different conceptions of the nature of mathematics and of mathematical knowledge. This paper offers an exposition of tenor of the arguments of five of these major schools of thought in the philosophy of mathematics: Logicism, Intuitionism, Empiricism, Formalism, and Psychologism.

Some Identified Paradoxes in Mathematical knowledge

Mathematics as a science of quantities or their representations focuses on the study of the numerical properties of the universe of “being” as presentations or representations. It is an area of knowledge that includes topics of numbers, formulae and related structures, shapes, and the spaces in which they are contained, and quantities and their changes.¹ Certainly, mathematics has played a great role in many parts of human knowledge. It is essential in the natural sciences, engineering, medicine, finance, computer science and the social sciences. Although mathematics is extensively used for modeling of representing phenomena in their quantitative terms, the fundamental truths of mathematics or mathematical knowledge are generally believed to be independent of any scientific experimentation.

¹. G. Kneebone. *Mathematical Logic and Foundations of Mathematics: An Introductory Survey*. [New York: Dover, 1963], 4.

With the turn of the twentieth century, there had been a profound activity on meta-mathematics geared towards greater research on the foundations of mathematics. In the process, several meta-mathematical issues converged to bring foundational problems in mathematics to the front burner. Prominent among these was the discovery of contradictions otherwise known as paradoxes in mathematical knowledge and assumptions, which ushered scepticism in the mathematical discipline and hence, reduced the objectivity and certainty of mathematical knowledge.² A paradox is a logically self-contradictory statement or a statement that runs contrary to one's expectation.³ It is a statement that, despite apparently valid reasoning from true premises, leads to a seemingly self-contradictory or a logically unacceptable conclusion.⁴ A paradox usually involves contradictory-yet-interrelated elements that exist simultaneously and persist over time.⁵ They result in "persistent contradiction between interdependent elements" leading to a lasting "unity of opposites".⁶ In logic, many paradoxes exist that are known to be invalid arguments, yet are nevertheless valuable in promoting critical thinking, while other paradoxes have revealed errors in definitions that were assumed to be rigorous, and have caused axioms of mathematics and logic to be re-examined.

Paradoxes form a natural object of philosophical investigation ever since the origins of rational thought; they have been invented as part of complex arguments and as tools for refuting philosophical theses (think of the celebrated paradoxes credited to Zeno of Elea, concerning motion, the continuum, the opposition between unity and plurality, or of the

-
- ² Yul Erchov and Palytus E. [*Mathematical Logic*. Moscow: Miv Publishers, 1984], 9.
 - ³ Andrea Cantini and Riccardo Bruni. "[Paradoxes and Contemporary Logic](#)". [Stanford Encyclopedia of Philosophy](#) (Fall 2017 ed.). Ed. Edward Zalta. (2017), 1.
 - ⁴ Thomas Bolander. "[Self-Reference](#)". *The Metaphysics Research Lab, Stanford University*. 2013, p.2.
 - ⁵ W. Smith and M. Lewis. "Toward a theory of paradox: A dynamic equilibrium model of organizing". *Academy of Management Review* 36, no. 2 (2011): 381.
 - ⁶ Jonathan Schach, Marianne Lewis, Sebastian Raisch, and Wendy Smith. "[Paradox Research in Management Science: Looking Back to Move Forward](#)". *Academy of Management Annals* 10, no. 1 (2016): 6

arguments entangling the notions of truth and vagueness, credited to the Megarian School, and Eubulides of Miletus). Paradoxes—termed as *Insolubilia*—formed also a substantial part of logical and philosophical investigations during the Middle Ages. However, between the end of the 19th century and the beginning of the 20th century, the foundations of logic and mathematics were affected by the discovery of a number of difficulties constituting paradoxes—involving *fundamental notions and basic methods of definition and inference*, which were usually accepted as unproblematic. The earliest modern paradoxes concerned the notions of ordinal and cardinal number, and the Russell's paradox.

With regard to the notions of ordinal and cardinal number, Burali-Forti, a mathematician of Peano's school, attempted to prove that the ordinal numbers are not linearly ordered.⁷ Assuming by contradiction that the class **ON** of all ordinals could be linearly ordered, he observed that then **ON** itself would be well-ordered and it would possess an ordinal $\Omega \in \mathbf{ON}$. Thus, **ON** would be similar (order-isomorphic) to a proper initial segment of itself, the one determined by Ω , contradicting a well-known theorem about well-ordered sets. The result was published in 1897 and, though Burali-Forti's original aim is impossible to achieve, his argument showed that the collection **ON** is problematic at best.

However, Cantor, the father of set theory, had noticed similar difficulties already in 1895, when he pointed out a problem involving the notion of the cardinal number and implying that one cannot consistently think of “the set of all conceivable sets”, say M . Were M a genuine set, then it would possess a cardinal number m , which would be the maximum cardinal number. But one could also consider the set $\wp(M)$ of all subsets of M , and by Cantor's theorem, the cardinality of $\wp(M)$ ought to be strictly bigger than the purported maximum m : contradiction. As a consequence, Cantor suggested a crucial distinction – still regarded as “subjective”, i.e., mathematically not precise, by Hilbert – between totalities that cannot be conceived as a whole (the *inconsistent* ones) and those which

⁷ Cantini, Andrea; Bruni, Riccardo "[Paradoxes and Contemporary Logic](#)". [Stanford Encyclopedia of Philosophy](#) (Fall 2017 ed.). Ed. Edward Zalta. (2017), 21.

can be regarded as completed. Roughly, the former is a collection that cannot be an element of other collections, whereas the latter is a small collection, which can be an element of other collections. This corresponds to the distinction between *classes* and *sets*, later made precise and axiomatized in the class-theoretic approach (von Neumann, Bernays, Gödel). In the case of the difficulty discovered by Burali-Forti, the consequence for Cantor was that the multiplicity of ordinal numbers is itself well-ordered, but is not a set: hence, no ordinal can be assigned to it, and the antinomy is resolved.

Russell's paradox, on the other hand, is the most famous of the logical or set-theoretical paradoxes, which Russell is said to have discovered in the late spring of 1901, while working on his *Principles of Mathematics* published in 1903.⁸ It arises within naïve set theory by considering the set of all sets that are not members of themselves. Such a set appears to be a member of itself if and only if it is not a member of itself. Hence, the paradox. In other words, Russell's paradox questions whether a "list of all lists that do not contain themselves" would include itself, and showed that attempts to found set theory on the identification of sets with properties or predicates were flawed.⁹ Thus, unlike Burali-Forti's paradox, Russell's paradox does not involve either ordinals or cardinals, relying instead only on the primitive notions of set and set inclusion. Central to any theory of sets is a statement of the conditions under which sets are formed. In addition to simply listing the members of a set, it was initially assumed that any well-defined condition (or precisely specified property) could be used to determine a set. For example, if *T* is the property of being a teacup, then the set, *S*, of all teacups might be defined as $S = \{x: T(x)\}$, the set of all individuals, *x*, such that *x* has the property of being *T*. Even a contradictory property might be used to determine a set.

⁸. Irvine, Andrew David and Harry Deutsch. "Russell's Paradox". *The Stanford Encyclopedia of Philosophy* Ed. Edward N. Zalta. (Spring 2021 Edition), 2.

⁹. Crossley, J.N.; Ash, C.J.; Brickhill, C.J.; Stillwell, J.C.; Williams, N.H. *What is Mathematical Logic?* [London-Oxford-New York: [Oxford University Press](https://www.oxforduniversitypress.com/), 1972], 60.

For example, the property of being both **T** and **not-T** would determine the empty set, the set having no members.¹⁰

More precisely, naïve set theory assumes the so-called naïve or unrestricted Comprehension Axiom, the axiom that for any formula $\phi(x)$ containing x as a free variable, there will exist the set $\{x: \phi(x)\}$ whose members are exactly those objects that satisfy $\phi(x)$. Thus, if the formula $\phi(x)$ stands for “ x is prime”, then $\{x: \phi(x)\}$ will be the set of prime numbers. If $\phi(x)$ stands for “ $\neg(x=x)$ ”, then $\{x: \phi(x)\}$ will be the empty set. But from the assumption of this axiom, Russell's contradiction follows. For example, if we let $\phi(x)$ stand for $x \notin x$ and let $R = \{x: \phi(x)\}$, then **R** is the set whose members are exactly those objects that are not members of themselves. Is **R** a member of itself? If it is, then it must satisfy the condition of not being a member of itself and so it is not. If it is not, then it must not satisfy the condition of not being a member of itself, and so it must be a member of itself. Since by classical logic one case or the other must hold – either **R** is a member of itself or it is not – it follows that the theory implies a contradiction. As Russell tells us, it was after he applied the same kind of reasoning found in Cantor's diagonal argument to a “supposed class of all imaginable objects” that he was led to the contradiction.¹¹

Standard responses to the paradox attempt to limit in some way the conditions under which sets are formed. The goal is usually both to eliminate **R** (and similar contradictory sets) and, at the same time, to retain all other sets needed for mathematics. This is often done by replacing the unrestricted Comprehension Axiom with the more restrictive Separation Axiom, namely the axiom that given any consistent set **S** and any formula $\phi(x)$ with x free, there will be a set $\{x \in S: \phi(x)\}$ whose members are exactly those members of **S** that satisfy $\phi(x)$.

¹⁰. Andrew Irvine and Harry Deutsch. "Russell's Paradox". *The Stanford Encyclopedia of Philosophy* Ed. Edward N. Zalta. (Spring 2021 Edition), 1.

¹¹. Bertrand Russell. *The Principles of Mathematics*. [Cambridge: Cambridge University Press, 1903], 136.

If we now let $\phi(x)$ stand for the formula $x \notin x$, it turns out that the corresponding set, $\{x \in S : x \notin x\}$ will not be contradictory since it consists only of those members found within S that are not members of themselves. Hence, the set fails to include itself.

The significance of Russell's paradox can be seen once it is realized that, using classical logic, all sentences follow from a contradiction. For example, assuming both P and $\neg P$, any arbitrary proposition, Q , can be proved as follows: from P we obtain $P \vee Q$ by the rule of Addition; then from $P \vee Q$ and $\neg P$ we obtain Q by the rule of Disjunctive Syllogism. Because set theory underlies all branches of mathematics, many people began to worry that the inconsistency of set theory would mean that no mathematical proof could be completely trustworthy. Only by eliminating Russell's paradox could mathematics as a whole regain its consistency. The discovery of these paradoxes in mathematical assumptions not only ushered scepticism in the mathematical discipline and hence, reduced the objectivity and certainty of mathematical knowledge, it also created a storm in philosophy of mathematics. While seeking to resolve these paradoxes therefore, philosophers proposed several schools of thought regarding the foundations and nature of mathematical knowledge, considered below.

Logicism

In the philosophy of mathematics, Logicism is the school of thought with the view that mathematical truths are ultimately logical truths. Stephan korner explain logicism as the view that, "mathematics is an extension of logic, some or all of mathematics is reducible to logic, or some or all of mathematics may be modelled in logic."¹² For the logicians, mathematics can be known *a priori*, and our knowledge of mathematics is just part of our knowledge of logic in general, and is thus analytic, not requiring any special faculty of mathematical intuition. Based on this view, logic is considered the proper foundation of mathematics and all mathematical statements are said to be necessarily logical truths. According to Leon

¹² Stephan korner. *The Philosophy of Mathematics: An Introductory Essay*. [New York: Dover Publications, 2009],3.

Horsten, “the Logicians project consists essentially in attempting to reduce mathematics to logic”.¹³ Russell who shares in this conception of the foundations of mathematics, says that, “All mathematics is symbolic logic”. For Benjamin Pierce, “Mathematics is the science that draws necessary conclusions”¹⁴ (qtd. in Howard Eves, 150). Presenting the Logician thesis in two parts, Rudolf Carnap writes:

1. The concepts of mathematics can be derived from logical concepts through explicit definitions.
2. The theorems of mathematics can be derived from logical axioms through purely logical deductions.¹⁵

Historically, the idea that mathematics is logic in disguise goes back to Leibniz, who tried to establish a relationship between mathematics and logic. But an earnest attempt to carry out the Logicians' program in detail took place in the 19th century when the basic principles of central mathematical theories were articulated especially by Richard Dedekind and Giuseppe Peano and the principles of logic discovered by Gottlob Frege. Bertrand Russell and Alfred North Whitehead also championed this programme. However, Frege is accorded the status of the founder of Logicism. In his seminal work, *Basic Laws of Arithmetic*, he built up arithmetic from a system of logic with a principle of comprehension, which he called “Basic Law V” (For a concept F and G , the extension of F equals the extension of G , if and only if for all objects a , Fa if and only if Ga), a principle that he took to be an acceptable part of logic. The principle can be further represented as follows:

$$(x/Fx) = (x/Gx) \equiv$$

In other words, the set of the F s is identical with the set of G s if the F s are precisely the G s. However, Frege's construction was flawed by Bertrand

¹³. Leon Horsten, “Philosophy of Mathematics”. *Stanford Encyclopedia of Philosophy*. Sept. 25, 2007, 2.

¹⁴. Howard Eves. *Foundations and Fundamental Concepts in Mathematics*. [Boston: PWS-Kant, 1990], 150.

¹⁵. Rudolf Carnap. “The Logician Foundations of Mathematics”. *Benacerraf and Putnam*. Trans. E. Putnam and G. Massey. [New York: Dover, 1983], 41.

Russell in his famous Letter to Frege in 1902, where he showed that Frege's *Basic Law V* entails contradictions. Frege abandoned his logicist program after this, but it was continued by Russell and Whitehead, who attributed the paradox to “vicious circularity” and built up what they called “ramified type theory” to deal with it. Nevertheless, modern Logicians or neo-logicians such as Bob Hale, Crispin Wright, etc., have abandoned *Basic Law V* in favour of abstraction principle such as Hume's principle (the number of objects falling under the concept F equals the number of objects falling under the concept G if and only if the extension of F and the extension of G can be put into one-to-one correspondence. The number of the F s = the number of the G s $\equiv F \cong G$. Where $F \cong G$ means that the F s and the G s stand in one-to-one correspondence with each other. Frege required the Basic Law V to be able to give an explicit definition of the numbers, but all the properties of numbers can be derived from Hume's principle.

However, most philosophers of Mathematics today doubt that Hume's principle is a principle of logic. Crispin Wright, for instance, has in recent years sought to qualify this claim by arguing that “Hume's principle is analytic of our concept of numbers, and therefore at least of the laws of reason”.¹⁶ This shows the many problems that confront any version of logicism. For instance, how can one show that given the logicist account of the nature of numbers, *William Shakespeare* is not a number? More generally, how can one show on such an account that no number is a concrete individual? Again, can the logicist account for (i) how *natural* numbers may be applied in counting finite collections? And (ii) how *real* numbers may be applied in measuring continuously varying magnitudes such as lengths, periods of time, etc.? Furthermore, can logic commit one to the existence of anything, or any kind of thing? And, how does one show that the *natural* number n is the same abstract object of the *integer* n , the *rational* number n , and the *real* number n ? These are some of the epistemic problems facing the logicist school of thought in the philosophy of mathematics.

¹⁶ Ernst Snapper. “The Three Crises in Mathematics: Logicism, Intuitionism and Formalism”. *Mathematics Magazine*. 52, no.4(1979): 207.

Intuitionism

In the philosophy of Mathematics, Intuitionism originated from the works of the Mathematician L. E. J. Brouwer, who, according to Robert Audi, argues that “Mathematical objects arise from the *a priori* forms of the volitions that inform the perception of empirical objects”.¹⁷ From this springboard, Intuitionists seek to reconstruct what they consider to be the corrigible portion of Mathematics in accordance with Kantian concept of being, intuition and knowledge. Based on this impression, Intuitionism maintains the view that, “Mathematics is essentially an activity of construction”.¹⁸ It emphasizes the construction of ideas in the mind, such that, natural numbers, mathematical proofs and theorems are considered to be mental constructions. In other words, for the Intuitionists, mathematics is a free creation of the mind, and valid results are about what the mind creates (rather than about an external reality, as in platonism, or about nothing, as in formalism). From the standpoint of intuitionism, the basic criterion for truth of a mathematical reasoning is intuitive evidence of the possibility of performing a mental experiment related to this reasoning. Intuitionists have challenged many of the oldest principles of mathematics as being nonconstructive and hence mathematically meaningless. For instance, Intuitionistic mathematics rejects the set-theoretic approach to the definition of mathematical concepts, as well as ways of reasoning customary in classical logic. Intuitionism points out non-formal, but “intuitive” subjects, as fundamental for the foundation of mathematics. Ernst Snapper succinctly captured the Intuitionists' conception of the nature of mathematics when he wrote that, for the Intuitionists, “mathematics is mental activity, which consists in carrying one after another, those mental constructions which are inductive and effective”.¹⁹ In other words, by combining fundamental ideas, one reaches a definite result.

¹⁷. Robert Audi. *The Cambridge Dictionary of Philosophy*. [Cambridge: Cambridge University Press, 1999], 542.

¹⁸. Leon Horsten. “Philosophy of Mathematics”. *Stanford Encyclopedia of Philosophy*. Sept. 25, 2007, 2.

¹⁹. Ernst Snapper. “The Three Crises in Mathematics: Logicism, Intuitionism and Formalism”. *Mathematics Magazine*. 52, no. 4 (1979): 207

From their general consideration about the nature of Mathematics, Intuitionists find non-constructive existence proofs unacceptable. Non-constructive existence proofs are proofs that purport to demonstrate the existence of a mathematical entity having a certain property without even implicitly containing a method for generating an example of such an entity. Intuitionism rejects such concepts as “theological” and “metaphysical” and as incompatible with the nature of mathematics. The characteristic feature of non-constructive existence proofs is that they make essential use of the principle of excluded middle: $\phi \vee \neg \phi$ or one of its equivalents, such as the principle of double negation: $\neg \neg \phi \Rightarrow \phi$

In classical Aristotelian logic, these principles are valid and their proofs are majorly based upon contradiction. On the contrary, the Intuitionistic logic as postulated by Brouwer's student, Arend Heyting, is different from the classical logic because it does not contain the law of excluded middle and therefore frowns upon proofs by contradiction. Heyting lays stress on the fact that intuitionism is concerned with mental mathematical constructions, which require a different form of logic. The Intuitionist logic is obtained by removing the principle of excluded middle (and its equivalents) from classical logic. In the first decades of the 20th century, parts of the Mathematical community were sympathetic to the Intuitionists critique of classical mathematics and to the alternative that it proposed. This situation, however, changed when it became clear that in higher Mathematics, the Intuitionist alternative differs rather drastically from the classical theory, and its mathematical analysis was extremely complicated and very different from classical Mathematical analysis. This dampened the enthusiasm of the Mathematical community for the Intuitionistic project, although followers of Brouwer have continued to develop intuitionistic Mathematics onto the present day.

However, intuitionistic mathematics does not appear to be an intellectually attractive school than the classical mathematics because it is disastrously weak, relying upon a deductively puny logic. In the first instance, it does not appear correct to hold that with the process of construction of mathematical object or knowledge is extra-linguistic or

mental. Rather, it is *real*, like a process executable in a computer. Thus, *constructive objects* are not to be understood as mentally conceived objects, but concrete objects, like the letters of an alphabet, that is a (finite or infinite) collection of discernible signs. Brouwer's assumption that certain objects of mathematics and mathematical operations are sufficiently *evident*, so that manipulation of these objects by such operations cannot lead to inconsistencies—assumption that is grounded on Kant's concept of intuition (*Anschauung*) – are objectionable and mere images of free mathematical creation. As intuitionism, lacks one of the principal virtues of a successful mathematical theory that yields enough consequences of the right sorts. Of course, there is a range of 'right' sort of conventional consequences that the intuitionist cannot obtain. Those are the intuitionistic equivalents of the *tertium non datur* or TND, $\varphi \vee \neg\varphi$. Conventional mathematicians deem TND mathematically essential. Without this assumption, one cannot prove such mainstays of conventional real analysis as the Intermediate Value Theorem: that the graph of a continuous function whose value is strictly positive at the left end of a nontrivial closed interval and strictly negative at its right end has to cross the x-axis at least once in the interval. Hence, the omission of TND drags intuitionism below the level of acceptable theory.

Empiricism

Empiricism is a form of mathematical realism that denies that mathematical knowledge can be known *a priori* at all. Generally, mathematical realism is a position which maintains that mathematical entities exist independently of the human mind. Thus, humans do not invent mathematics, but rather, discover it. In line with this realist position, the Empiricist school of thought in the philosophy of mathematics maintains that “we discover mathematical facts by empirical research just like facts in any of the other sciences”. In line with the Empiricists thesis, Max Tegmark said that “not only do all mathematical objects exist, but nothing else does. All structures that exist mathematically also exist physically”.²⁰

²⁰. Max Tegmark. “The Mathematical Universe”. *Foundations of Physics* 38, no. 2 (2008):101.

Empiricism is one of the classical positions advocated in the early 20th century, but primarily it arose in the middle of the century. An important early proponent of this view was John Stuart Mill. However, contemporary mathematical empiricism especially as formulated by W. Quine and Putman is basically supported by what is called “Indispensability argument”. According to this argument, mathematics is indispensable to all empirical sciences, and if we want to believe in the reality of the phenomena described by the sciences, we ought to believe in the reality of these entities required for this description. For instance, since physics needs to talk about numbers in offering any of its explanations, then numbers must exist. Thus, Quine and Putman's argument is basically naturalistic – centering in the existence of mind-independent mathematical entities as the best explanation for experience. Putman strongly rejected the term “Platonists” as implying an over-specific ontology that was not necessary to mathematical practice in any real sense. He advocated a form of “pure realism” that rejected much of Quasi-empiricism in mathematics. However, empiricism in mathematics raises some epistemic problems, which militates against its success in establishing objectivity of mathematical of mathematical knowledge. A fundamental problem that may be raised here is: If mathematics is just as empirical as the other sciences, then this suggests that mathematical knowledge or assumptions are just as fallible as those of the other science, and just as contingent.

Formalism

As a school of thought in the philosophy of Mathematics, formalism is an anti-realist position which denies both physical and mental meanings to mathematics. It makes the symbols and rules the object of study and conceives mathematics essentially as “the manipulation of the meaningless symbols of a first-order language, according to explicit syntactic rules”.²¹ Thus, for the Formalists, mathematical statements may be thought of as “statements about the consequences of certain string manipulation rules”. For example, in the “game” of Euclidean geometry

²¹. Ernst Snapper. “The Three Crises in Mathematics: Logicism, Intuitionism and Formalism”. *Mathematics Magazine*. 52, no. 4 (1979): 207

(which is seen as consisting of some strings called “Axioms” and some “rules of inference” to generate new strings from given ones), one can generate the string corresponding to such theorems as Pythagorean theorem, etc.

According to formalism, a mathematical statement is true if and only if it is derivable in the system. But in a fundamental level, all mathematical systems are on a par. There can be at most programmatic reasons for preferring one system over another. Inconsistent system must be modified. A version of formalism known as deductivism holds that mathematical theorem (e.g. Pythagorean theory) are not absolute but relative. It is argued that if one assigns meaning to strings in such a way that the rules of the game become true (i.e. true statements are assigned to the axioms and the rules of inference are truth-preserving), then one must accept the theorem, or rather the interpretation one has given it must be a true statement. This is believed to be true for all mathematical statements.

A major early proponent of formalism was David Hilbert, whose program was intended to be a complete and consistent axiomatization of all of mathematics.²² Hilbert aimed to show the consistency of mathematical system from the assumption that the “finitary arithmetic” (a subsystem of the usual arithmetic of the positive integers, chosen to be philosophically uncontroversial) as consistent. Hilbert's goal of creating a system of mathematics that is both complete and consistent were dealt a fatal blow by Godel's *Incompleteness theorem*, which states that sufficient expressive consistent axiom systems can never prove their own consistency.²³ Thus, in order to show that any axiomatic system of mathematics is in fact consistent, one needs to first assume the consistency of a system of mathematics that is in a sense stronger than the system to be proved consistent. Other formalists such as Rudolf Carnap,

^{22.} Richard Zach. “Hilbert's Program”. *The Stanford Encyclopedia of Philosophy*. Ed. Edward Zalta. (Summer 2003). Metaphysics Research Lb., Stanford University.

^{23.} Martin Kusch. “Psychologism”. *The Stanford Encyclopedia of Philosophy*. (Edward Zalta (ed.). 2020, n.p.

Alfred Tarski, and Haskell Curry, considered mathematics to be the investigation of the formal axiom systems. However, what I might consider the main criticism against formalism is that the actual mathematical ideas that occupy mathematics are far removed from the string manipulation games mentioned above. Formalism is thus silent on the question of which axiom systems ought to be studied, as none is more meaningful than other from a formalistic point of view.

Psychologism

Generally, psychologism is a philosophical position, according to which psychology plays a central role in grounding or explaining some other, non-psychological type or fact or law. The Oxford English Dictionary defines this term as, “The view or doctrine that a theory of psychology or idea forms the basis of an account of metaphysics, epistemology or meaning: The explanation or derivation of mathematical or logical laws in terms of psychological fact”. The word, “Psychologism” entered the English language as a translation of the German word, “psychologismus”, a term coined by the Hegelian Johann Eduard Erdmann in 1870 to critically characterize the philosophical positions of Eduard Beneke.

In the philosophy of mathematics, psychologism is the view that “mathematical concepts and/or truths are grounded in, derived from or explained by psychological facts or laws”.²⁴ In other words, for this school of thought, mathematical objects are the products of mental processes or at least depend on mental processes. According to J. Shorupski, John Stuart Mill was the initial “advocate of a type of logical psychologism”.²⁵ Psychologism can perfectly be traced back even to the British empiricism, especially to Locke's ideational epistemology. It can also be gleaned from Descartes Cogito ergo sum as well as from Kant's transcendentalism.

This school of thought in the philosophy of mathematics was famously

^{24.} S. Gallagher. *Phenomenology*. [New York: Palgrave Macmillan, 2012], 20.

^{25.} Shorupski. *Mill: Argument of the Philosophers*. [New York: Routledge, 2010], 127-128.

criticized by Frege in his *The Foundations of Arithmetic* and many of his works and essays including his review of Husserl's *Philosophy of Arithmetic*. Edmund Husserl, in the first volume of his logical investigations, called *The Prolegomena of Pure Logic*, criticized psychologism thoroughly and sought to distance himself from it. However, an outstanding criticism that can be raised against psychologism is that it conceives mathematics in a context of discovery in rather than in a context of justification. It concerns itself more with the source of mathematical knowledge in our psychology rather than the justification of the objectivity of mathematical truths or assumption, whereas mathematical truths are not reducible to our psychology. Thus, psychologism fails to do justice to the idea of the objective nature of mathematical truths; for, it is precisely because such truths are objective and eternal that mathematical and logical laws cannot be laws about states of affairs (be they mental or physical).

Conclusion

The diverse positions of the five major schools of thought in the philosophy of mathematics: Logicism, formalism, intuitionism, empiricism and psychologism – were explored in this paper. It goes without saying that each of these views provide salient insights into the nature of mathematics and the source of mathematical knowledge. However, their obvious weaknesses highlighted herein show forth their defeasible status, and ultimately calls for a deeper interest in the philosophy of mathematics. Perhaps, with the evolution of knowledge there is the prospect that greater certainty about the foundations and nature of mathematics can be gained.

Bibliography

- Audi, Robert.1999. *The Cambridge Dictionary of Philosophy*. Cambridge: Cambridge University Press.
- Bolander, Thomas. 2013. "Self-Reference". "Paradoxes and Contemporary Logic". Stanford Encyclopedia of Philosophy (Fall 2017 ed.). ed. Zalta, Edward N.<https://plato.stanford.edu/archives/fall2011/enries.self-defense/>. Retrieved: 23 March, 2023.
- Cantini, Andrea; Riccardo Bruni. 2017. "Paradoxes and Contemporary

- Logic". Stanford Encyclopedia of Philosophy (Fall 2017 ed.). ed. Zalta, Edward N. <https://plato.stanford.edu/archives/fall2011/entries/contemporary-logic/>. Retrieved: 4 February, 2023.
- Crossley, J., C. Ash, C. Brickhill, J. Stillwell, and H. Williams. 1972. *What is Mathematical Logic?* Oxford: Oxford University Press.
- Erchov, Yul and Palytus E. 1984. *Mathematical Logic*. Moscow: Miv Publishers.
- Eves, Howard. 1990. *Foundations and Fundamental Concepts in Mathematics*. Boston: PWS-Kant.
- Horsten, Leon. "Philosophy of Mathematics". *The Stanford Encyclopedia of Philosophy*. (Spring 2021 Edition). Eds. Edward Zalta and Uri Nodelman. <https://plato.stanford.edu/archives/spr2003/entries/mathematics/>. Retrieved: 23 Sept., 2022.
- Gallagher, S. *Phenomenology*. New York: Palgrave Macmillan, 2012.
- Irvine, Andrew David and Harry Deutsch. "Russell's Paradox". *The Stanford Encyclopedia of Philosophy* (Spring 2021 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/spr2021/entries/russell-paradox/>>. Retrieved: 3 Feb., 2024.
- Kneebone, G. 1963. *Mathematical Logic and Foundations of Mathematics: An Introductory Survey*. New York: Dover.
- Kusch, Martin. "Psychologism". *The Stanford Encyclopedia of Philosophy*. Ed. Edward Zalta 2020. <https://plato.stanford.edu/archives/spr2020/entries/psychologism/>.
- Oxford English Dictionary. 2003. England: Pearson Educational Press.
- Russell, Bertrand. 1903. *The Principles of Mathematics*. Cambridge: Cambridge University Press.
- _____. 1919. *Introduction to Mathematical Philosophy*. London: George Allen and Unwin and New York: The Macmillan Co.
- Schad, Jonathan; Marianne Lewis; Sebastian Raisch; and Wendy Smith. 2016. "Paradox Research in Management Science: Looking Back to Move Forward". *Academy of Management Annals*. 10 (1): 5–64
- Shorupski, J. 2010. *Mill: Argument of the Philosophers*. New York: Routledge.
- Skorner, Stephan. 2009. *Philosophy of Mathematics*. New York: Dover Publications.
- Smith, W.; and M. Lewis. 2011. "Toward a Theory of Paradox: A Dynamic Equilibrium Model of Organizing". *Academy of*

Management Review. 36 (2): 381–403.

Snapper, Ernst. 1979. “The Three Crises in Mathematics: Logicism, Intuitionism and Formalism”. *Mathematics Magazine*. 52(4): 207-216.

Tegmark, Max. 2008. “The Mathematical Universe”. *Foundations of Physics*. 38(2): 101-150.

Wright, C. 1983. “Frege's Conception of Numbers as objects”. *Scot Philosophical Monographs*. Vol. 2 Aberdeen: Aberdeen University Press.

Zach, Richard.(2003). “Hilbert's Program”. The Stanford Encyclopedia of Philosophy. Ed. Edward Zalta. Metaphysics Research Lab., Stanford University. <https://philpapers.org/references> . Retrieved: 5 Jan., 2023.