

What has Philosophy got to do with Mathematics?

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Abstract

This paper explores the philosopher's interest in mathematics. While it is a statement of fact that philosophers and mathematicians, for the most part, agree with the thesis that there exists some connection between these disciplines, it remains to show what type of relationship exists between them. The paper argues that while philosophy assumes the second-order function in its relations to mathematics and questions the basic assumptions, axioms, and theorems of mathematics, there is a sense in which philosophy does not tell the mathematician what to do or not. We contend that after exposing the low points of the basic assumptions and axioms of mathematics, the onus lies exclusively on the mathematicians to either make changes to their system or remain sacrosanct. In the final third, the paper appraises the three basic schools in the philosophy of mathematics—logicism, formalism, and intuitionism. The paper finds out that, though these schools are seen as grounding for mathematics, Gödel's incompleteness theorem suggests that none of them is philosophically viable. Thus, we conclude that, regardless of the shortcomings and the inability of the aforementioned schools to successfully provide the foundation for mathematics, these schools have helped raise questions and sustained discourses in the philosophy of mathematics.

Keywords: Philosophy, Mathematics, Logicism, Formalism, Intuitionism, Philosophy of Mathematics.

Introduction

The two main abstract of all academic disciplines in human history are, undoubtedly, philosophy and mathematics. That there exist some close connections between the two disciplines is a fact that is admissible to both philosophers and mathematicians alike. To this end, historical facts, shows that philosophers have had a special as well as a kin attraction to the enterprise of mathematics. Shapiro corroborated the above when he intimated

relating to Plato that “the entrance of Plato academy is said to have been marked with the phrase 'let no one ignorant of geometry enter here'”.¹ This position, as one may argue is hinged on the imperativeness of training in mathematics as a spring board for grasping and making sense of the universe. Besides Plato, Aristotle considers abstraction as the process through which mathematics arises after rejecting his master's idea of forms.

From Descartes to Leibniz, from Pascal to Bolzano, from Russell to Whitehead, from Hilbert to Brouwer, from the most celebrated Frege to Peirce, from Church to Gödel to Wittgenstein, one finds, with profundity, mathematicians that delved into the enterprise of philosophy. In fact, it is better to understand such a thinker as Descartes first as a mathematician before the foundational element of his philosophy will make sense; it is also better to understand Ludwig Wittgenstein as an aeronautic engineer before one can fully grasp the extent of his impact and contributions to philosophy. While some mathematicians came to philosophy with the aim of using it as a guide to their activities in the field of philosophy, philosophers are attracted to mathematics with the reflective intent of both grounding mathematics and making the doing of mathematics better.

This paper seeks to answer the question of the interface between philosophy and mathematics; to do this, the paper proceeds from a preliminary discourse on what philosophy and mathematics are about to a discourse on the philosopher's interest in mathematics and the foundations of the philosophy of mathematics. It then proceeds, in the final third, from a discourse on the three basic schools of the philosophy of mathematics being logicism, formalism, and intuitionism to an appraisal of the tenets of these schools.

Philosophy and Mathematics: What are they about?

In this section, we attempt a discourse on the definition and the

¹. Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*. (New York: Oxford University Press, 2000). 3.

nature, amongst others, of philosophy and mathematics. This paper does not discuss, in large volume, the core of philosophy and mathematics; the reason for the above is the fact that such a large and voluminous discourse on mathematics and philosophy would take a book and not a small article. There is a sense in which stating the above in the opening lines of this section is imperative.

Philosophy defies any universally accepted definition.² It has been defined variously by different thinkers in the history of its study. The absence of a universally accepted definition made Sidney Hook to skip the question of the definition of philosophy in his article "The Use of Philosophy." This, in his thinking, is hinged on the fact that any definition that is advanced will allude to some kind of conception of philosophy that will most likely not be accepted by all philosophers. While this may be considered embarrassing in some quarters and hence, taken to constitute a

^{2.} Godfrey Ozumba argues against beginning the discourse on the meaning of philosophy this way, that is, by stating that there is no universally accepted definition of philosophy. According to him, "when a businessman begins the presentation of his wares by announcing the doubtfulness (dubiety) of the quality of his wares, then such a businessman should not expect to go home with any profit. His avowals may be a display of nobility and sincerity of character, but the best he can get is eviction from the business world" (see his article entitled "What is Philosophy?" in Ozumba, G. O. and Ojong, K.A. (Ed.) *The Mirror of Philosophy*, pg. 16 for a more lucid discourse on what pattern the presentation of the answer to the question of what is philosophy should take). However, this paper holds and elevates the nobility and sincerity of character over and against acceptance into the business world temporarily and the immediate gains it brings. We believe that the sincerity and nobility of character can position the discipline as unique and sustain it in the business world of intellectual investigation for a longer and enduring period of time than the crave for immediate acceptance and approval.

drawback to the discipline,³ we consider it both unique and the beauty of the enterprise of philosophy.

The above notwithstanding, philosophy etymologically is derived from two Greek words, *philo* connoting love and *sophia* connoting wisdom. Taken together, it means “the love of wisdom.” Away from the etymological definition, philosophers have also advanced varied definitions of philosophy. While it is considered essentially a reflective as well as a critical activity, Staniland defines it as the critical evaluation of the ideas we live by. For Dewey, philosophy is thinking that has become conscious of itself and is the criticism of criticisms.⁴ For C.E.M. Joad as cited in Ozumba, philosophy is the clearinghouse to which the results of all other human intellectual endeavors are brought in, sifted assessed, and evaluated.⁵ It remains to state here that philosophy, through metaphysical reflections, seeks to understand the nature of reality in order to create meaning, order, and stability in our commerce with the world. Through critical epistemology, philosophy questions the means by which the specialized sciences arrive at truth or trustworthy knowledge of things. Through ethics, philosophy tackles the problem of how man should act to achieve mental, emotional, and spiritual stability in his interaction with his fellowman and environment. Hence, philosophy is a

3. We find people who have not really had an interface with or studied philosophy, commenting on it and drawing implications from it. When people consider philosophy as being embarrassingly indefinable, it is the most puzzling description of all. If they really believe that philosophy is indefinable and that the apparent lack of a definition that is accepted by all philosophers is a drawback to the discipline, it is because philosophers have not really done a good job at unveiling and defending the uniqueness of the enterprise of philosophy. It also tells the professional philosophers, more and more, that in writing and teaching, they have to be bringing in discourses on the concept and nature of philosophy as well as reasons why the discipline is true with such an enthusiasm of monumental proportion. This apologetics is very important in times like this.

4. Dewey, J. *Democracy and Education*. (New York: Macmillan, 1916), 381.

5. Ozumba, G. O *Introduction to Philosophy of Language*. (Ibadan: Hope Publishers, 2004), 18.

“meaning, order, and stability-seeking enterprise by the critical questioning of the truth of being, of whatever is.”⁶

Within the context of discourse in this paper, philosophy is a fluid discipline which has no definite boundaries. It pries into other disciplines as a philosophy of this and that. Philosophy has tentacles spreading to different areas of human intellectual endeavors and plays the role a gadfly over the rest of the disciplines.

Mathematics has been said to occupy a unique place in human inquiry. It is the most arduous and certain of all the sciences and it plays a key role in most, if not all scientific work. It is based on the above assertion that the C.F. Gauss (1777-1855) declared mathematics to be the “queen of the sciences”.⁷ Tymoczko seem to be thinking in the above light when he stressed that “pure mathematics stands at the pinnacle of rational thought”.⁸ For Dummett, “mathematics presents itself as a science in the general sense in which history is a science, namely, as a sector in the quest for truth”.⁹ Colyvan further stressed regarding mathematical results and the importance of mathematics thus:

The results and methods of mathematics are often surprising and elegant, occasionally revealing an austere, abstract beauty more typically found in the arts. Mathematics permeates our intellectual life and has helped to shape modern society. Science is inconceivable apart from mathematics, and we often measure the rigor or 'hardness' of a science by the amount of mathematics it employs. We rely on mathematics when we build bridges, fly airplanes, use computers or get cash from automatic tellers.¹⁰

^{6.} Unah, J. I “Philosophy and Society” Unah J. I (Ed.) *Philosophy, Society and Anthropology*. (Lagos: Fedec Publishers, 2002), 9-10.

^{7.} Colyvan, M. *An Introduction to the Philosophy of Mathematics*. (Sydney: University of Sydney Press, 2011), 1.

^{8.} Tymoczko, T. (Ed.) *New Directions in the Philosophy of Mathematics: An Anthology*. (Basel: Birkhauser, 1985), Pg. xiii

^{9.} Dummett, M. “What is Mathematics About?” *Mathematics and Mind*. Alexander George (Ed.). (Oxford: Oxford University Press, 1994), 19.

^{10.} Colyvan, M. *An Introduction to the Philosophy of Mathematics*, xiii.

Mathematicians study the properties of natural numbers, real numbers, ordinal numbers, groups, topological space, differential manifolds, lattices and the likes.¹¹ Colyvan seems to agree with Dummett but takes it further when he intimated that, mathematics studies mathematical entities such as numbers, sets, functions and the structural relationship between them. These entities, should they exist, are very peculiar, abstract, they do not have spatio-temporal locations and do not have casual powers.¹²

In the same vein, Wang argued that certainty, abstractness, precision, its broad range of application and its dry beauty are the most impressive features of mathematics, he further argued that the “precision and certainty (of mathematics) are to a large extent due to the abstractness which also in part explains the wide applicability. But the close connection to the physical world is an essential feature which separates mathematics from mere game or symbols. Mathematics coincides with all that is exact in science”.¹³ Brown corroborated the fact of the certainty of mathematics when he averred that mathematics is “the one and only place where we humans can be absolutely sure we got it right”.¹⁴ Deducible from the above are the features of mathematics, chief amongst which are its certitude and exactitude.

Contributing to discourse on the concerns and nature of mathematics, Tiles argued that, “mathematics deals with things that do not change. Moreover, it can only yield principles that concerns relations between things, because numbers are assigned only after selection of a unit of measure; and geometry deals only with shapes, which in turn are characterized by the relations

^{11.} Dummett, M. “What is Mathematics About?”, 19.

^{12.} Colyvan, M. *An Introduction to the Philosophy of Mathematics*, 1.

^{13.} Wang, H “Theories and Practice in Mathematics” Thomas Tymoczko (Ed.) *New Directions in the Philosophy of Mathematics: An Anthology*. (Basel: Birkhauser, 1985), 143.

^{14.} Brown, J. R. *Philosophy of Mathematics: A Contemporary Introduction to the World of Proofs and Pictures* (2nd Edition). (New York: Routledge, 2008), 2.

between their parts, and with spatial relation between shapes".¹⁵ What this translates into is that, mathematics cannot yield insights into the nature or the principle of change within the context of the individual as it were or into the casual principles that govern the universe.

It must also be noted that mathematics differs from the other sciences in terms of the methodology that drives its operations and activities. Methodologically, mathematics proceeds through a priori means using deductive proof, as opposed to the aposteriori methods of experimentation and induction employed and deployed by the other sciences.¹⁶ It must also be stated that mathematics is not revisable in the manner the other sciences are revisable. This is because, once a theorem is proven and accepted as true, it becomes axiomatised and stands forever.

The Philosopher's Interest in Mathematics and the Foundation of the Philosophy of Mathematics

To the query of whether there is a connection between philosophy and mathematics, in a rather sublime manner; one can argue that there is not so much direct and conscious connection between mainstream mathematics and mainstream philosophy. This is because, following from the division and specialization of areas of academic disciplines, philosophers and mathematicians do not easily understand or comprehend the research and findings of each other in such a straight forward manner. Be that as it may, as Shapiro argues, mathematics does not lie far from the concerns of such philosophical fields as epistemology, metaphysics, logic, cognitive sciences amongst others. Philosophy is also not far from the main concerns of such mathematical fields as logic, set theory, category theory, computability and geometry. In fact, logic is taught in both mathematics and philosophy departments all over

^{15.} Mary Tiles "Philosophy of Mathematics" in Nicholas Bunnin, E. P. Tsui-James (Ed.) *The Blackwell Companion to Philosophy, Second Edition.* (Malden, MA: Blackwell Publishers Ltd, 2003), 351.

^{16.} Colyvan, M. *An Introduction to the Philosophy of Mathematics*, 1.

the world.¹⁷

Shapiro further advances three reasons for the fascination of philosophy with mathematics. The first reason according to him is that both (mathematics and philosophy) are the first rational attempts to comprehend the world and both were either born or underwent transformation in ancient Greece. The second reason, according to Shapiro for this connection is the fact that mathematics is a vital case study for the philosopher. Here, the philosophers are concerned with matters of epistemology, ontology, semantics and logic and the philosopher is concerned with the question of normativity.¹⁸ The third reason as he argued lies in epistemology (the study or the theory of knowledge). Mathematics is very important because of its central place in every scientific effort at understanding the physical world. In fact, all scientific domains of study and the most part of engineering have mathematics as a pre-requisite.¹⁹

In the same vein, the sciences cannot do without mathematics and it is in that context that mathematics has been argued to be “the queen of all science”. Drawing from the above assertion that mathematics is instrumental to all human efforts at conceptualizing and understanding the material world, one can argue that the “philosophy of mathematics is a branch of general epistemology, and that mathematics is an important case for general epistemology and metaphysics.”²⁰

In history, philosophers and mathematicians as Shapiro further argued believed that philosophical matters such as ontology and metaphysics determined the proper practice of mathematics. This accounts for why Plato held that the subject matter of mathematics is eternal, unchanging, and locatable in the ideal realm. Mathematical objects like numbers and geometric objects are created and destroyed and they cannot be changed.²¹ In

^{17.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 4.

^{18.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 5.

^{19.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 6.

^{20.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 6.

^{21.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 7.

the same vein, Ijiomah avers that with regards to vision, defense and attainment of reasonableness, the aggregate of all applications of mathematics cannot compare with the theory or the philosophy of the discipline. Put differently, there cannot be a proper application of mathematics without an assumption of a theoretical base from which meanings of propositions and concepts used in mathematics can be evaluated.²² Quoting J.J Ross, Ijiomah further argued that the reality of anything is not factual until it acquires significance for a certain theory and such a theory needs to be a moral one and its guide human actions in any disciplinary matrix.²³

What that means therefore is that practice without a moral and conceptual base is not only blind as Ijiomah argues, but also dehumanizes. It is based on the above logic that the introduction of foundational studies, that is, the philosophy of X which includes the philosophy of mathematics is legitimized.²⁴ What then is the philosophy of mathematics and what is its history and nature?.

According to Shaipro, the philosophy of mathematics belongs to a genre that includes the philosophy of physics, biology, psychology, language, philosophy of logic, and even philosophy of philosophy.

The theme is to deal with philosophical questions that concerns an academic discipline, issues about the metaphysics, epistemology, semantics, logic and methodology of the discipline. The philosophy of X is pursued by those and only those who care about X and wants to illuminate its place in the overall intellectual enterprise.²⁵

22. Ijiomah, C. O “What is Philosophy of Mathematics?” Ozumba, G.O and Ojong, K. A (Ed.) *The Mirror of Philosophy*. (Uyo: El-Johns Publisher, 2014), 437.

23. Ijiomah, C. O “What is Philosophy of Mathematics?”, 437.

24. Ijiomah, C. O “What is Philosophy of Mathematics?”, 437.

25. Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, vii.

Shapiro in his article “Philosophy of Mathematics and its Logic: Introduction” further showed that for any field of study X, the main purposes of the philosophy of X are to interpret X. The philosopher of mathematics immediately encounters sweeping issues, typically concerning all of mathematics. Most of these questions come from general philosophy: matters of ontology, epistemology, and logic. What if anything, is mathematics about? How is mathematics pursued? Do we know mathematics and if so, how do we know mathematics? What is the methodology of mathematics? To what extent is this methodology reliable? What is the proper logic of mathematics? To what extent are the principles of mathematics objective and independent of the mind, language and social structure of mathematicians?²⁶ Any attempt to answer these plethora of questions, situates one within the realm of the sub-field of philosophy designated as the philosophy of mathematics. Consequently, someone who practices the philosophy of mathematics,

should gain something by adopting a philosophy of mathematics, that is, an appreciation of the discipline, an orientation towards it, a vision of its role in understanding the world. The philosopher of mathematics needs to say something about mathematics itself, something about the human mathematician, and something about the world where mathematics gets applied.²⁷

It must be noted that mathematicians for the most part, are concerned with as well as engaged in filling two kinds of gaps, not always sharply distinguishable: - gaps consisting in the absence of theorem within existing theories, and gaps consisting in the absence of theories. Philosophy comes to play in the attempt at filling theses gaps. According to Stephen Korner, philosophical considerations are more likely to be influential when the task is not

^{26.} Shapiro, S “Philosophy of Mathematics and its Logic: Introduction”, 4.

^{27.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, vii-viii.

so much as to find theorem, as to find theories.²⁸ He further argued that:

They (philosophical considerations) will be more influential in the construction of theories intended to provide the 'foundation' of mathematics, than in theories providing the mathematical apparatus, say, for a particular branch of physics. That at least, the originators of logicist, formalist, and intuitionist mathematics have been strongly influenced by philosophical assumptions, insight, prejudices (whatever one may call them) cannot be doubted if we take their own statements seriously.²⁹

It must also be noted that the various diagnoses that a sound mathematic must be deduced from logical principles (logicism) or that it must be a formalism whose consistency is proved by finite methods (formalism) or that it must consist of reports on intuitive constructions (intuitionism), and so forth are all philosophical diagnoses.³⁰ These are discourses ramifying the three basic 'isms' of the philosophy of mathematics and each "leads to a programme and its implementation by a mathematical theory. If the theory is found to be unsatisfiable, it is abandoned or modified. But two or more incompatible programmes may all be satisfiable and their abandonment or resurrection may be due to philosophical arguments or even fashion.³¹ Herein again, lies the answer to the question of what philosophy has got to do with mathematics.

It is also incumbent on this attempt at showing the relationship between philosophy and mathematics to argue following from Korner that both philosophy and mathematics are at work in any reconstruction process of mathematical theories. To quote Korner:

The replacement analysis, or reconstruction of

²⁸. Korner, S. *The Philosophy of Mathematics: An Introductory Essay*. (London: Hutchinson Publishers, 1971), 183.

²⁹. Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 183.

³⁰. Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 183.

³¹. Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 185-186.

mathematical theories in the field of 'foundation of mathematics' have thus, been a joint task of mathematicians and philosophers. The defence of satisfiable programmes or of programmes not known unsatisfiable proceeds largely by philosophical or to use a much abused word, metaphysical argument. The implementation or attempted implementation of a programme on the other hand is a piece of mathematics.³²

What is crystal clear and perhaps, undeniable from the above quotation is the fact that while the task of the defence as well as the justification of programmes (satisfiable or unsatisfiable) lies within the purview of philosophical considerations and arguments, the task of the implementation of the programmes lies exclusively within the realm of mathematics. Tiles corroborated the above in a rather sublime manner when she averred that there is "no sharp line where mathematics ends and philosophy begins. Indeed, one of the striking features of the development of mathematics in the twentieth century was the way in which attempts to solve philosophical and conceptual problems generated new mathematics".³³ To this end, a mathematician who adopts a philosophy of mathematics should gain something by this, "an orientation towards the work, some insights into the role of mathematics, and at least a tentative guide to the direction of mathematics, what sort of problems are important? What questions should be posed? What methodologies are relevant? What is likely to succeed? and so on".³⁴ Put differently, the philosophy of mathematics is that branch of philosophy that is saddled with the responsibility of understanding mathematics. Hence,

It investigates the limits of mathematics, the subject matter of mathematic, the relationship between mathematics and the rest of the sciences, the logic of mathematical proofs, and the significance of the language of mathematics to

^{32.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 186.

^{33.} Mary Tiles "Philosophy of Mathematics", 346.

^{34.} Shapiro, S. *Philosophy of Mathematics and Its Logic*, 6.

mathematical practice. These are all important topics and... are significant for both philosophy and mathematics.³⁵

In the above context, philosophy sheds light to mathematics and hence, it can be argued that even the very “methodology of mathematics hangs on the answers to some of the philosophical questions that impose themselves upon us”.³⁶ For Thomas Tymoczko, “the philosophy of mathematics, begins when we ask for a general account of mathematics, a synoptic vision of the discipline that reveals its essential features and explains just how it is that human beings are able to do mathematics”.³⁷ Its business is to provide the foundation of mathematics. Philosophy as it were is kept in business by the fact that there are competing alternatives to the issue of foundations. What philosophy does is to adjudicate amongst the competition, evaluating the conflicting claims. The instrument of adjudication which is also the instrument that was used in generating the competition in the first place is mathematical logic.³⁸

The philosophy of mathematics can also be viewed as “the working philosophy of the professional mathematician, the philosophical attitude towards his work that is assumed by the researcher, teacher or user of mathematics”.³⁹ It is within this context that Ijiomah averred that as a foundational study, the philosophy of mathematics employs a high level search for an irreducible base or genesis on which the structure of mathematical knowledge lies.⁴⁰ It seeks to take a closer look at the

³⁵. Colyvan, M. *An Introduction to the Philosophy of Mathematics*, 2.

³⁶. Colyvan, M. *An Introduction to the Philosophy of Mathematics*, 2.

³⁷. Tymoczko, T. (Ed.) *New Directions in the Philosophy of Mathematics: An Anthology*. Viii.

³⁸. Tymoczko, T. (Ed.) *New Directions in the Philosophy of Mathematics: An Anthology*, xv.

³⁹. Hersh, R. “Some Proposals for Reviving the Philosophy of Mathematics” Thomas Tymoczko (Ed.) *New Directions in the Philosophy of Mathematics: An Anthology*. (Basel: Birkhauser, 1985), 10.

⁴⁰. Ijiomah, C. O “What is Philosophy of Mathematics?”, 437.

assumptions/foundations of mathematics with the intent of validating them and the relevant part of which is to make such foundations as solid as possible.

The above notwithstanding, there is a sense in which philosophy in general or the philosophy of mathematics in particular do not relate or interfere with mathematics and that is in the fact that philosophy does not spell out technically, what mathematics should or should not do. Ijiomah seems to be thinking in that light when he avers:

Philosophy of mathematics cannot tell mathematicians what to do or what not to do in the discipline of mathematics. It can only hold a mathematician responsible for the formal or logical consequences or implications of his axiom, definitions, theories etc. This is the job of philosophy of mathematics. After exposing their implications, it is left for mathematics to continue or make changes or make adjustments in his belief or practice.⁴¹

In the same vein, “just as the philosophy of law does not legislate, or the philosophy of science devices or test scientific hypotheses; so, we must realize that the philosophy of mathematics does not add to the number of mathematical theorems or theories. This is because, it is not mathematics, it is rather a reflection on mathematics”.⁴² The above quotation vividly clarifies the scope and limit of the philosophy of mathematics. While it does not dictate and lay down the curricula that should guide the mathematician in their enterprise, it holds mathematicians responsible and liable for the consequences of their basic assumptions and findings after laying bare the flaws and low points of their activities leaving the mathematicians the latitude to either attempt reforms and amendments to their earlier positions or not.

Regarding the beginning of the philosophy of mathematics, Ijiomah, quoting Burns and Bell intimated that if we retreat into

^{41.} Ijiomah, C. O “What is Philosophy of Mathematics?”, 443.

^{42.} Körner, S. *The Philosophy of Mathematics: An Introductory Essay*, 9.

the past, we can realize that philosophy of mathematics is relatively old in that the Egyptians of the pre-dynasty period (5000-4000BC) in Ahmose papyrus reflected upon mathematics, Plato and Kant also conceptualized on mathematics but in terms of maturation, we can say that the philosophy of mathematics is relatively new in the sense that it was from the twentieth century that philosophers of mathematics introduced a symbolically, logical and rigorous approach to conceptualization in mathematics.⁴³ Tracing the origin or the introduction of the philosophy of mathematics to the nineteenth century, Tiles argued that the introduction to “the philosophy of mathematics often begins where Korner's (1960) influential introduction began, outlining three position; namely logicism, formalism and intuitionism. These were the three contending schools that emerged from nineteenth century mathematical move aimed at providing rigorous foundation for mathematical analysis”.⁴⁴

While a comprehensive and exhaustive discourse on these schools lies beyond the purview of this paper, there is a sense in which it is incumbent on us to at least, in brief, engage in a discourse on the three schools (the basic 'isms') that have been used to ground mathematics by the philosophy of mathematics. This will be done in the paragraphs that follow.

Exploring Logicism, Formalism and Intuitionism as the Basic Schools in the Philosophy of Mathematics

In the opening lines of his article “Philosophy of Mathematics in the Twentieth Century” Michael Detlefsen, argued that philosophy of mathematics in the twentieth century has largely been molded by three major influences. First, is the works of Immanuel Kant and secondly, the response to Kant's idea of geometry in nineteenth-century that was focused on the discovery of non-Euclidean geometries. The third influence is the new discoveries in logic that emerged with growing swiftness

⁴³. Ijjomah, C. O “What is Philosophy of Mathematics?”, 438.
⁴⁴. Mary Tiles “*Philosophy of Mathematics*”, 345.

and force during the closing half of the nineteenth century.⁴⁵ He further averred:

In one way or another, the main currents of twentieth-century philosophy of mathematics and, in particular, the so-called logicist, intuitionist and formalist movements are all attempts to reconcile Kant's revolutionary plan for mathematical epistemology with the equally revolutionary ideas of Gauss, Bolyai and Lobatchevsky in geometry, and the powerful ideas and techniques developed by Boole, Peirce, Peano, Frege and other nineteenth-century figures in logic.⁴⁶

It is from these profound influences that logicism, formalism and intuitionism draw their impetus as we will understand from what follows.

i. Logicism

Logicism is one of the three broad schools under which the philosophy of mathematics is studied. It is the “thesis that mathematics, or at least some significant portion of mathematics, is part of logic”.⁴⁷ This thesis is the conjunction of two theses: expressibility logicism: mathematical propositions are (or are alternative expressions of) purely logical propositions; and derivational logicism: the axioms and theorems of mathematics can be derived from pure logic.⁴⁸ For Shapiro the idea that the concepts and objects of mathematics like number can be defined from logical standpoint; and drawing from these definitions, the theorems of mathematics are derivable from the principle of logic

^{45.} Detlefsen, M. “Philosophy of Mathematics in the Twentieth Century” Staurt G. Shanker (Ed.) *Routledge History of Philosophy (Vol. IX): Philosophy of Science, Logic and Mathematics in the 20th Century*. (London: Routledge, 2004), 50.

^{46.} Detlefsen, M. “Philosophy of Mathematics in the Twentieth Century”, 50.

^{47.} Harold Hodes “Logicism” Robert Audi (Ed.) *The Cambridge Dictionary of Philosophy* (2nd Edition). (Cambridge: Cambridge University Press, 1999), 550.

^{48.} Harold Hodes “Logicism”, 550.

is what is referred to as logicism.⁴⁹ The doctrine of logicism as Nino Cocchiaella argues can be aptly stated in the following two fold claim: (a) that all the concepts of classical mathematics are explicitly definable in terms of purely logical concepts; and (b) that all of the theorems of classical mathematics can be derived from the laws of logic through purely logical deductions.⁵⁰ Be that as it may, it must be noted that logicism does not presuppose the reducibility of mathematics to set theory; conversely, it is the reducibility of mathematics in its classical form to the concepts and laws of logic.

What can be deciphered from the attempts at clarifying the concept of logicism by Harold Hordes, Stewart Shapiro and Nino Cocchiaella as presented in the above lines is that within the context of modern scholarship where logic has become more mathematical and mathematics has become more logical is that logicism presupposes and precludes the fact that all of mathematics or some of it is part of logic.

Within the context of logic in the 19th century, logicism has been argued to be a “philosophical theory whose time has come”.⁵¹ The concept of logicism owes its arrival and representation to Gottlob Frege (1848-1925). This argument is based on the fact that “prior to Frege, logicism was just such a concept whose pure form was obscured by irrelevant accretions; and in his life's work it was Frege who first presented this concept to humanity in its pure form and developed it as a doctrine of the first rank”.⁵² The “classical” presentations of logicism were Frege's *Grundgesetze der Arithmetik* and Russell and Whitehead's

^{49.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 108.

^{50.} Cocchiaella, Nino “Frege, Russell and Logicism: A Logical Reconstruction” Haaparanta Leila and Hintikka Jaakko (Ed.) *Frege Synthesized: Essays on the Philosophical and Foundational Works of Gottlob Frege*. (Boston: D. Reidel Publishing, 1986). 198.

^{51.} Cocchiaella, Nino “Frege, Russell and Logicism: A Logical Reconstruction”, 197.

^{52.} Cocchiaella, Nino “Frege, Russell and Logicism: A Logical Reconstruction”, 197.

Principia Mathematica. Frege took logic to be a formalized fragment of second order logic supplemented by an operator forming singular terms from “incomplete” expressions, such a term standing for an extension of the “incomplete” expression standing for a concept of level 1 (that is, type 1).⁵³

The introduction of the concept script differing from Leibniz's calculus philosophicus ratiocinator is a pointer to the intent of the development of a logistic programme. It must also be noted that it is difficult to say with exactitude when Frege first conceived his logicist project of reducing the mathematical truths to those of logic, but it may have been as early as the time of his Habilitation. Frege's first major work in logic, *Begriffsschrift* (literally “Concept-script,”), published in 1879, is more or less a primer in the logical system he endorsed as an alternative to the then-dominant Boolean algebraist logics.⁵⁴

Following from the above, it feels safe to argue that while the *Begriffsschrift* as published in 1879 can be said to be the first attempt by Frege to develop his *Begriffsschrift* (concept script) laying out the basic notations and hinting on how the notations can be used to demonstrate the truth of arithmetic, Klement^b held that,

Frege went on to explain his logicist project in his 1884 *Die Grundlagen der Arithmetik* (The Foundations of Arithmetic) and further development of his formal logical work was put off until the publication of the first volume of his magnum opus, *Grundgesetze der Arithmetik* (Basic Laws of Arithmetic) in 1893.⁵⁵

While it must be noted that the *Grundgesetze* itself was devoted largely to the actual demonstration of what Frege considers as the most important principle in arithmetic beginning with only logical

^{53.} Cocchiaella, Nino “Frege, Russell and Logicism: A Logical Reconstruction”, 197.

^{54.} Klement K. C. “Gottlob Frege” Dean Moyar (Ed.) *The Routledge Companion to Nineteenth Century Philosophy*. (Abingdon: Routledge, 2010), 859.

^{55.} Klement K. C. *Frege and the Logic of Sense and Reference*. (New York: Routledge, 2002), 6.

axioms and inference rules, it added to the logic of *Begriffsschrift*, notations and axioms dealing with classes and extensions of concept necessary for, among other things, defining numbers in Frege's approach. These additions as Klement argued were "discovered to make the system of the *Grundgesetze* inconsistent by Russell in 1902".⁵⁶

Klement corroborated the above when he argued for the innovative temper of the *Begriffsschrift of the Grundgesetze*. He further showed that when Frege was at the verge of successfully creating a logical calculus (from volume I and II and to finish it in the third volume) with which he intends to take his logicist project to the final end, Bertrand Russell wrote to Frege in the June of 1902 while the second volume was still in the press. In his letter, Russell commended the ingenuity of Frege while showing at the same time, the apparent contradiction of the paradox of classes that was later to be referred to as Russell's paradox.⁵⁷

This contradiction stems from the basic law V of Frege's laws of arithmetic. Russell commended Frege for the project and stressed the fact that he agrees with Frege in all essentials, particularly his rejection of the psychological elements in logic and his development of the *Begriffsschrift* for the foundation of mathematics and formal logic. However, Russell points out the double correlation thesis, the contradiction of basic law V which was later to become known as Russell's paradox.

In his commentary on Russell's paradox, Ijiomah avers that what Russell is saying is that to assume that function, otherwise referred to as predicate can also comprise of indeterminate elements implies that it is possible to allow predication of properties to themselves.⁵⁸ This is a universal statement and

⁵⁶. Klement K. C. "Gottlob Frege" Dean Moyer (Ed.) *The Routledge Companion to Nineteenth Century Philosophy*, 860.

⁵⁷. Klement K. C. *Frege and the Logic of Sense and Reference*, 7.

⁵⁸. See Chris O. Ijiomah's unpublished manuscript of the book *Philosophy of Mathematics: An Introduction*, 90. We acknowledge and appreciate the contributions of the Late Distinguished Professor Chris Okezie Ijiomah who gladly availed us access to his unpublished work shortly before his demise from this world.

Russell seems to harp on this to make his point and show the contradiction. For Russell, as Ijiomah argues, to make such a universal propositional statement amounts to a contradiction and that while some can be predicated on themselves, others cannot.

Ijiomah further used predicate logic symbolization to show that the position of Frege yields an apparent contradiction and hence Frege got himself into trouble.⁵⁹ Frege, hit by the apparent contradiction in his system responded to Russell in a temper of academic and intellectual humility stressing that it has shaken the basis and foundations of his arithmetic while showing that the discovery of Russell is remarkable and will lead to great advances in logic.

Frege hastily prepared an appendix to the volume discussing the contradiction and offering as very tentative proposal for revision.⁶⁰ In the appendix of the *Basic Laws of Arithmetic*, Frege averred that “hardly anything more unwelcome can befall a scientific writer than that one of the foundations of his edifice shaken after the work is finished. This is the position into which I was put by a letter from Mr. Bertrand Russell as the printing of this volume was nearing completion”.⁶¹

Frege's immediate reaction to the paradox as one can see from the appendix was to attempt a modification of his principle regarding value ranges so as to maintain the general structure of his proof while regaining consistency. He eventually came to realize as Patricia Blanchette argue that “no such fix would work and that no principle governing values ranges could count both as purely logical by its own light, and sufficiently rich to ground the existence of an infinite collection of objects, i.e the numbers”.⁶² After all these attempts to recover from the blow, “probably, due to

^{59.} Ijiomah, C. O. *Philosophy of Mathematics: An Introduction*, 91-92.

^{60.} Klement K. C. “Gottlob Frege” Dean Moyer (Ed.) *The Routledge Companion to Nineteenth Century Philosophy*, 860.

^{61.} Frege, G. *Begriffsschrift*. “Begriffsschrift: Selections (Preface and Part I).” In M. Beaney, ed., *The Frege Reader*. (Oxford: Blackwell, 1879), 253.

^{62.} Blanchette, P. A. “Gottlob Frege” M. Forster and K. Gjesdal (Ed.) *The Oxford Handbook of Nineteenth-Century German Philosophy*. (Oxford: Oxford University Press 2015), 15.

some combination of poor health, the early loss of his wife in 1905, frustration from his failure to find an adequate solution to Russell's paradox, and disappointment over the continual poor reception of his works, Frege seems to have lost his intellectual steam. He produced almost no published work between 1906 and his retirement in 1918 and the initially planned third volume of the *Grundgesetze* never appeared".⁶³ Other major proponents of logicism whose systems cannot be accounted for in this paper for want of space are Richard Dedekind and Bertrand Russell.

ii. Formalism

As earlier stated, formalism, along with logicism and intuitionism are the three classical philosophical programme for grounding mathematics. According to Colyvan, formalism "takes mathematical notations and its manipulations to be the core business of mathematics".⁶⁴ In its purest form, Colyvan avers that formalism is the thesis that mathematics is nothing more than the manipulations of meaningless symbols.⁶⁵ For Shapiro, "the various philosophies that go by the name formalism pursues a claim that the essence of mathematics is the manipulation of characters".⁶⁶ He further argues that the list of the characters and allowed rules all but exhaust what there is to say about a given branch of mathematics. Hence, mathematics for the formalist does not go beyond typographical characters and rules for manipulating them.⁶⁷

The term 'formalism' according to Sinaceur may have at least three different meanings. First, 'formalism' can be understood as referring to a mathematical way of operating. A formalist way of doing mathematics shows how one can get new

⁶³. Klement K. C. "Gottlob Frege" Dean Moyer (Ed.) *The Routledge Companion to Nineteenth Century Philosophy*, Pg. 860.

⁶⁴. Colyvan, M. *An Introduction to the Philosophy of Mathematics*, 4.

⁶⁵. Colyvan, M. *An Introduction to the Philosophy of Mathematics*, 4.

⁶⁶. Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 140

⁶⁷. Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 140

and innovative results from the mere inspection of symbolic expressions used or coined for mathematical entities or properties. In a second meaning, 'formalism' means a philosophical attitude, which seeks an answer not to the question: how can one do mathematics in a general and very efficient way? But to the question: how or on what to ground mathematical practice? The third and more specific sense of 'formalism' comes from Hilbert's metamathematics, which combines logical analysis of mathematical procedures with philosophical views on the foundations of mathematical practice. This third sense is linked to Hilbert's concern with formal systems of mathematical theories, to his syntactic study of mathematical proof, and, notably, to his search for consistency proofs which would secure the soundness of mathematical reasoning against the paradoxes of set theory and would permit to avoid restrictions on classical logic.⁶⁸ It is within the context of the third meaning that formalism is considered synonymous with the works of Hilbert.

Formalism was created by the German mathematician, David Hilbert (1862-1943) in 1910.⁶⁹ In his contribution to the discourse on what constitutes formalism, J. M. Bochenski averred that:

Formalism is thus, a method which consists in completely disregarding the eidetic meaning of signs and operating with them on the basis of certain transformation-rules concerned only with the written shape of the signs. They are treated as though they were not signs but pieces in a game, which can be combined and transposed in various ways. It has therefore been said in jest that anyone who makes use of formalism does not know what he is saying, nor whether what he is saying is true.⁷⁰

⁶⁸. Sinaceur, H. B. "Taski's Practice and Philosophy: Between Formalism and Pragmaticism" Lindstrom, S. *et'al* (Ed.) *Logicism, Intuitionism and Formalism*. (Cham, Switzerland: Springer, 2009), 355-356.

⁶⁹. Snapper, E "The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism" in *Mathematics Magazine*. Vol. 52., No. 4, (Sept. 1979), 212.

⁷⁰. Bochenski, J.M *The Methods of Contemporary Thought* (Bern: D. Reidel Publishing Company, 1965), 40.

Put differently, formalism according to Bouwer is the viewpoint which maintains that human reason does not have at its disposal exact images either of straight lines or of numbers larger than ten, for example; and that therefore these mathematical entities do not have existence in our conception of nature any more than in nature itself. It is true that from certain relations among mathematical entities, which we assume as axioms, we deduce other relations according to fixed laws, in the conviction that in this way we derive truths from truths by logical reasoning. But this non-mathematical conviction of truth or legitimacy has no exactness whatever and is nothing but a vague sensation of delight arising from the knowledge of the efficacy of the projection into nature of these relations and laws of reasoning.⁷¹

He further averred that,

For the formalist, therefore, mathematical exactness consists merely in the method of developing the series of relations, and is independent of the significance one might want to give to the relations or the entities which they relate. And for the consistent formalist these meaningless series of relations to which mathematics are reduced have mathematical existence only when they have been represented in spoken or written language together with the mathematical-logical laws upon which their development depends, thus forming what is called symbolic logic.⁷²

It must however be noted that formalism as Shapiro argues has a better pedigree and acceptability amongst mathematicians than amongst philosophers of mathematics. That mathematicians have had occasion to introduce symbols which, at the time, seems to have no clear interpretation is a statement of fact that is corroborated by history.⁷³ To quote Shapiro,

^{71.} Brouwer, L. E. J. "Consciousness, Philosophy and Mathematics" in **Paul Benacerraf (Ed.) *Philosophy of Mathematics: Selected Readings*. Second Edition, New York: Cambridge University Press, 1983.** 78.

^{72.} Brouwer, L. E. J. "Consciousness, Philosophy and Mathematics", 78-79.

^{73.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 141

The very names 'negative numbers', 'irrational numbers', 'transcendental numbers', 'imaginary numbers', and 'ideal point at affinity' indicates ambivalence. Fortunately, the profession of mathematics has had its share of bold, imaginative souls, but it seems that some special folk provides the names. Although the newly introduced 'entities' proved useful for applications within mathematics and science, in their philosophical moments some mathematicians did not know what to make of them. What are imaginary numbers, really? A common response to such dilemmas is to retreat to formalism.⁷⁴

What should be noted from the above discourse on the concept of formalism is that, it is one of the big three, that is, the major schools in the study of the philosophy of mathematics which was systematized in the works of Hilbert. It is a method of studying mathematics or its foundations that completely disregards signs and symbols while considering the manipulation of characters as the very essence of mathematics.

Various classifications of the school of the philosophy of mathematics that goes by the name formalism have been advanced. Stewart Shapiro in his book *Thinking about Mathematics: The Philosophy of Mathematics*, distinguished term from game formalism. However, Alan Weir in his article "Formalism in the Philosophy of Mathematics" gave the breakdown of the forms in which formalism unveils itself to include game and term formalism, tractarian formalism (relating to Ludwig Wittgenstein's *Tractatus*), formalism and the positivists, and nominalist formalism.

iii. Intuitionism

Intuitionism started with the doctoral dissertation of L.E.J Brouwer (1881- 1966), Brouwer was a Dutch mathematician

⁷⁴. Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 141.

of extraordinary scope, vision and imagination and he was a major architect of twentieth century mathematics.⁷⁵ He conceives logicism and formalism as both focusing on the external trappings of mathematical communication and completely ignoring the essence of mathematics,⁷⁶ and this accounts for why he rejected them with a vehemence of monumental proportion.

In his article “Consciousness, Philosophy and Mathematics”, Brouwer avails that the long belief in the universal validity of the principle of the excluded third in mathematics is considered by intuitionism as a phenomenon of history of civilization of the same kind as the old-time belief in the rationality of π in the rotation of the firmament on as axis passing through the earth. And intuitionism tries to explain the long persistence of this dogma by two facts: first, the obvious non-contradictoriness of the principle for an arbitrary single assertion; secondly, the practical validity of the whole of classical logic for an extensive group of simple everyday phenomena.⁷⁷ In the closing lines of the same paper, Brouwer announced that “intuitionist mathematics is inner architecture, and that research in foundations of mathematics is an inner inquiry with revealing and liberating consequences, also in non-mathematical domains of thought”.⁷⁸ Just like Kant, Brouwer according to John Bell believed that mathematical concepts are admissible only if they are sufficiently grounded in intuition, he believed that mathematical theories are significant if and only if they concern entities which are constructed out of something given immediately

⁷⁵. Shapiro, S. “Philosophy of Mathematic and its Logic: Introduction” in Shapiro, S. (Ed.) *The Oxford Handbook of Philosophy of Mathematics and Logic*, 319.

⁷⁶. Shapiro, *Thinking about Mathematics: The Philosophy of Mathematics*, Pg. 185.

⁷⁷. Brouwer “Consciousness, Philosophy and Mathematics”, Pg. 94.

⁷⁸. Brouwer, “Consciousness, Philosophy and Mathematics”, 96.

by intuition, that mathematical definition must always be constructed, and that the complete infinite is to be rejected.⁷⁹ It is in the above sense that Bouwer was most Kantian of the three leading proponents of the three basic 'isms' of the philosophy of mathematics and himself referred to Kant's philosophy as "an old form of intuitionism".⁸⁰

Intuitionism as a school in the study of the philosophy of mathematics takes proofs seriously. It holds that proofs and constructions are all there is; intuitionism is sometimes called constructivism for this reason. Accordingly, mathematics is not taken to be about some pre-existing realm of mathematical objects. Intuitionism has it that mathematical objects needs to be constructed before one can sensibly speak about them.⁸¹ At the beginning of mathematics for the intuitionists, is their explanation of what the natural number 1,2,3 etc means. In this context, Snapper stressed that in, intuitionistic philosophy,

all human beings have a primordial intuition for the natural numbers within them. This means in the first place that we have an immediate certainty as to what is meant by number 1 and, secondly, that the mental process which goes into the formation of number 1 can be repeated. When we repeat it, we obtain the concept of number 2; when we repeat it again, the concept of number 3. In this way, human beings can construct any finite initial segment 1, 2, 3 etc for any natural number n. this mental construction of one natural number after the other would never have been possible if we did not have an awareness of time within us.⁸²

For intuitionist philosophy, mathematics should be defined as a mental activity and not as a set of theorems. It is the activity which consists in carrying out; one after the other, those mental

^{79.} Bell, J. *The Philosophy of Mathematics*. (N.P., N.D), 15-16.

^{80.} Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 174.

^{81.} Colyvan, M. *An Introduction to the Philosophy of Mathematics*, 7.

^{82.} Snapper, E "The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism", 210.

constructions which are inductive and effective in the sense in which the intuitionist construction of natural number is inductive and effective.⁸³ Korner corroborated the above when he averred that the subject matter of intuitionist mathematics is intuited, non-perceptual objects and constructions which are introspectively self-evident.⁸⁴ For Arend Heyting, a leading member of the intuitionist school as quoted by Bell,

Intuitionistic mathematics consists... in mental constructions; a mathematical theorem expresses a purely empirical fact, namely the success of certain construction '2+2=3+1' must be read as an abbreviation for the statement "I have effected the mental construction indicated in '2+2' and '3+1' and I have found that they lead to the same result."⁸⁵

The consequences of accepting this intuitionist definition of mathematics as effective and constructive are in two fronts: the first is that all intuitionist proofs, theorems, definitions etc are entirely constructive; secondly, mathematics cannot be reduced to any other sciences such as logic⁸⁶ as logicians have attempted to do. Herein lays the difference between intuitionism and logicism.

Intuitionism in the early part of the twentieth century was dominated, as Detlefsen argues, by two ideas; the first is the idea that what the mind brings forth purely of itself cannot be hidden from it; and the second is the belief that the existence of non-Euclidean geometries reveals important epistemological differences between geometry and arithmetic.⁸⁷ Beside Brouwer and Heyting, other prominent intuitionists are Poincare, Weyl and Dummett.

⁸³. Snapper, E "The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism", 210.

⁸⁴. Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 120.

⁸⁵. Bell, J. *The Philosophy of Mathematics*, 17.

⁸⁶. Snapper, E "The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism", 211.

⁸⁷. Detlefsen, M. "Philosophy of Mathematics in the Twentieth Century" Staurt G. Shanker (Ed.) *Routledge History of Philosophy (Vol. IX): Philosophy of Science, Logic and Mathematics in the 20th Century*. (London: Routledge, 2004), 70-71

As we draw to the conclusion of the discourse on the three basic 'isms' of the philosophy of mathematics, two points are imperative and hence, they are here stated. First, it must be noted that logicism, formalism and intuitionism cannot be held to provide a complete account of what constitutes the nature of mathematics. However, they all separately give expression to an important partial truth about the nature of mathematics; while mathematical truth and logical demonstration goes together in logicism, mathematical activities proceed by/from the performance of mental construction in intuitionism and the results of these construction are presented/evidenced through symbols in formalism. The second point is the argument that there is a sense in which logicism, formalism and intuitionism interrelate with each other. While both logicians and formalist formalized the various branches of mathematics, though for entirely different reasons (the logicist wants to use such formalization to show that the branch of mathematics in question belongs to logic; the formalist wants to use it to prove mathematically that, that branch is free of contradiction),⁸⁸ the codification of intuitionist mathematics can be shown, under certain interpretations, to be isomorphic with formalist systems.⁸⁹ It is in this sense that Korner argued that:

One frequently finds a desire to combine intuitive intensions with formalist precision. As a consequence of this mutual interaction, the sharp division of mathematicians and philosophers into logicians, formalists, and intuitionists, which was never quite realistic, except for the protagonist of these movements, is likely to lose much of its values and become little more than a pedagogical device.⁹⁰

A Critique of the Three Basic 'Isms' of the Philosophy of Mathematics

A plethora of criticisms have been advanced against the three

⁸⁸. Snapper, E "The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism", 214.

⁸⁹. Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 149.

⁹⁰. Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 150.

schools of the philosophical reflections about mathematics. Within the context of formalism, the German philosopher and mathematician, who also systematized logicism and contributed to the development of quantification theory, Gottlob Frege is the most ardent and effective critic of formalism. In his *Foundations of Arithmetic*, more specifically section 92-103 which is titled "other numbers", took a swipe on the thinkers who introduced new numbers as a way of providing solutions to equations which are hitherto and previously insoluble. In this regard, Frege was unimpressed and argued that introducing new signs as is done in formalism to do new things is inadmissible since they could be introduced to perform contradictory functions.⁹¹

In a follow up to the above, Frege, while contrasting the formalist position with his logicist position criticized the formalist definition of numbers as either circular in presupposing the consistency of what is defined, which supposes the sign's signify something after all, or else as impotent to secure the truth of the propositions that formal manipulations are supposed to underwrite.⁹² In this sense, one can assert that Frege shows that the formalists are superficial in their attitude since as it is clear, they did not offer a formal theory of positive integers.⁹³

There are however, two main objections that Frege set out against formalism. The first is the problem of its applicability. To quote Weir, "if mathematics is just a calculus in which we shuffle uninterpreted symbols, then why has it been applied so successfully and in so many ways, to so many different things, ordinary physical objects, sub-atomic objects, fields, properties, and indeed, from one part of mathematics?"⁹⁴ In the same vein,

⁹¹. Simons, Peter, 'Formalism' in Andrew D. Irvine (ed.), *Philosophy of Mathematics*, (Amsterdam: North Holland, 2009), 5.

⁹². Simons, Peter, 'Formalism' in Andrew D. Irvine (ed.), *Philosophy of Mathematics*, 6.

⁹³. Simons, Peter, 'Formalism' in Andrew D. Irvine (ed.), *Philosophy of Mathematics*, 6.

⁹⁴. See Weir, Alan, "Formalism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Fall 2019 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/fall2019/entries/formalism-mathematics/>>. Retrieved on the 20/05/2020.

Shapiro asserts the game formalist is left with a daunting problem, why are the mathematical games so useful in the sciences considering the fact that mathematics is the queen of all sciences? Why should one think that the meaningless games of mathematics should have an application? It is a truism that it does have application and hence, the need to explain these applications. This also goes within the context of the application of mathematics within mathematics and brings to the fore, the question of why is the game of complex analysis useful in the game of real analysis or arithmetic? This is all the more troubling and problematic for someone who is a game formalist or is sympathetic to game formalism or formalism in general.⁹⁵

Frege's response is that it is applicability and applicability alone that elevates mathematics from a game to a science. To quote Frege with regards to his criticism of game formalism,

An arithmetic without thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable. Why can arithmetic equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? It is applicability alone which elevates arithmetic from a game to the rank of a science.⁹⁶

The second objection is hinged on the problem of metatheory. The problem this raises for the formalist is that metatheory is itself a significant piece of mathematics, supposedly committed to an infinite realm of objects which are not, on the face of it, concrete.

⁹⁵ Shapiro, S. *Thinking about Mathematics: The Philosophy of Mathematics*, 146.

⁹⁶ Frege, G. *Begriffsschrift*. "Begriffsschrift: Selections (Preface and Part I)." In M. Beaney, ed., *The Frege Reader*, Pg. 147.

Symbols of the expressions of the object language game calculus may be finite-ink marks and the likes; but since there are infinitely many expressions, theorem and proofs, these themselves must be taken to be abstract types. At best, the formalist can achieve no more than a reduction in commitment from the transfinite realms of some mathematical theories.⁹⁷

Brouwer contributed to the discourse on the critique of Hilbert's formalism. Having taking formalism as antithetical to his intuitionism, he averred that the formalist conception recognizes no other mathematics than the mathematical language and it considers it essential to draw up definitions and axiom and to deduce from these other propositions by means of logical principles which are also explicitly formulated beforehand.⁹⁸ In his critique of formalism, Bouwer argued that “the formalist purported foundations of mathematical laws on the axiomatic method are nothing but mere linguistic explanations, devoid of content, which, as such, do not really explain anything”.⁹⁹ Bouwer further rejected formalism not just as a mathematical way but as a philosophy or more accurately as mathematical project to solve philosophical problem.¹⁰⁰ It must be noted that this rejection is in favour of his intuitionism and that is why he made no distinction between formalism and logicism (the other two schools in philosophy of mathematics) mentioning Russell, Cantor, Dedekind, Zermelo amongst others as formalist. It must also be noted that one of the differences between Bouwer and Hilbert is the fact that Hilbert denied a foundational status to intuition which is the basis of Bouwer's intuitionism. It remains to note that

^{97.} See Weir, Alan, "Formalism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Fall 2019 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/fall2019/entries/formalism-mathematics/>>. Retrieved on the 20/05/2020.

^{98.} Sinaceur, H. B. “Taski's Practice and Philosophy: Between Formalism and Pragmaticism”, 358.

^{99.} Brouwer, L. E. “Intuitionism and formalism” in **Paul Benacerraf (Ed.)** *Philosophy of Mathematics: Selected Readings. Second Edition*, (New York: Cambridge University Press, 1983), 359.

^{100.} Brouwer, L. E. “Intuitionism and formalism”, 359-360.

the Hilbert programme of establishing a one-one correspondence between all propositions of classical mathematics on the one hand and formulae embodying them on the other hand has been shown by Gödel to be impossible, or at least, highly problematic.¹⁰¹

Hilbert however took the criticisms of Bouwer seriously and published a series of articles to respond to him. In these articles, as Sinaceur argues, Hilbert introduced a new mathematics, namely, metamathematics, and developed technically and philosophically, his famous finitistic program where he incorporated what he referred to as the constructivity principle and some other intuistic insight in a more systematic and radical manner.¹⁰² To quote Sinaceur:

He considered that one must have something primitive and irreducible to begin with. He then changed his mind about intuition and logic and accepted to give intuition a basic role in the formal treatment. From 1922 onwards, he gave up Frege's and Dedekind's idea to provide for arithmetic a foundation that would be independent of all intuition and experience and he claimed that as a condition for the use of logical inference and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought.¹⁰³

What can be deduced from the above is that Hilbert admittedly agreed that mathematicians starts with an intuitive notion of number and confirmed Bouwer's account of formalism and went further after understanding insights as physical perception and formalism as mechanistic operating with some signs, formulas and arrays. The latter are indeed, according to his new point of view, the concrete and surveyable objects of metamathematics which

^{101.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 100.

^{102.} Sinaceur, H. B. "Taski's Practice and Philosophy: Between Formalism and Pragmaticism", 363-364.

^{103.} Sinaceur, H. B. "Taski's Practice and Philosophy: Between Formalism and Pragmaticism", 364

according to Hilbert is the continual theory of formalized proof.¹⁰⁴ The leading idea in this theory of proof according to Johann Von Neumann is that even if the statements of classical mathematics should turn out to be false as to content, nevertheless, classical mathematics involves an internally closed procedure which operates according to fixed rules known to all mathematicians and which consist basically in constructing successively certain combinations of primitive symbols which are considered “correct” or “proved”.¹⁰⁵

In his contribution to discourse on the critique of formalism, Stefan Kahrs avers that formalism has been criticized as being hypocritical, as lacking history and as being dogmatic.¹⁰⁶ Regarding its lack of history, Imre Lakatos showed that formalism disconnect the history of mathematics, since, according to the formalist concept of mathematics, there is no history of mathematics proper.¹⁰⁷ While this criticism is both arguable and unfair, it remains to argue that Lakatos conceives the formalist as a soul that is irredeemably lost. Regarding the conception of formalism as being dogmatic, Lakatos held formalism to be the latest dogmatic philosophy in mathematics.¹⁰⁸ This argument stems from the outright misunderstanding and misconception of formalism- this is because as Stefan argues, formalism is skeptical philosophy and is not a dogmatic philosophy. In fact, it has its background and foundation in the mistrust and lack of faith in the capabilities of human beings.

Against intuitionistic mathematics, it has been argued that “if one were to restrict mathematical constructions to those which

^{104.} Sinaceur, H. B. “Taski's Practice and Philosophy: Between Formalism and Pragmaticism”, 365.

^{105.} Neumann, J. V “The Formalist Foundations of Mathematics” in **Paul Benacerraf (Ed.) *Philosophy of Mathematics: Selected Readings. Second Edition, (New York: Cambridge University Press, 1983),*** 61-62.

^{106.} Stefan Kahrs “A Formalist's Perspective of Mathematics”, (N.P. N.P, 1999), 10.

^{107.} See Imre Lakatos quoted in Stefan Kahrs “A Formalist's Perspective of Mathematics”, 10.

^{108.} Stefan Kahrs “A Formalist's Perspective of Mathematics”, 11.

can be recorded without the intuitionist negation and double negation, intuitionist mathematics would be greatly improvised, without even being secured against the possibility of conflicting reports. Such security as Korner argues could not be achieved unless a safe class of constructions could be delimited.¹⁰⁹ It must also be noted that while it seems possible to discern within intuitionism, a hard core of constructions which can actually be carried out on perceptual objects, the report of this as Korner intimates would then be “theorems of a strictly finitist mathematics. But here, we come up against a further difficulty, namely, that the self-evident construction takes place in intuition and not in sense-perception”.¹¹⁰

For Weyl, in his contributions to the critique of intuitionism within the context of mathematics, intuitionist mathematics is too narrow to accommodate theoretical physics; perhaps, because of the high degree of arbitrariness involved.¹¹¹ It must also be made apparently clear that not everyone is ready to concede the existence of the faculty of intuition, which though different from sense-perception, apprehends particular objects as given. There are many philosophers who hold that introspection does not reveal the presence of any intuitive faculty of the Kantian and Bouwerian type. To deny this faculty is to deny Bouwer's positive view that potentially infinite, infinitely proceeding sequences exist, in the sense of being intuitively constructible.¹¹² The above notwithstanding, intuitionist mathematics in the sense of Bouwer's programme is most likely to continue to thrive and flourish regardless of its acceptance or otherwise. To quote Korner,

Many mathematicians are profoundly interested in its (intuitionist mathematics') problem without being noticeably interested in its privileged status. Belief in the satisfiability of the intuitionist programme has not been shaken. It is not any longer possible to deduce 'mathematics'

^{109.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 140.

^{110.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 140.

^{111.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 144.

^{112.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 148.

from 'logic' after the fashion of Frege; or to prove by Hilbert's finite methods that classical mathematics is consistent. It is still possible to pursue intuitionist mathematics as originally conceived.¹¹³

It is in the above sense that the hard-nosed intuitionist claims that intuitionism has been axiomatised and that it has given mathematics an entirely satisfactory foundation.¹¹⁴ In fact, intuitionist philosophy, particularly its insistence that mathematical existence is constructability, together with its rejection of the law of excluded middle and the actual infinities, has been of great influence in the development of both mathematics and the philosophy of mathematics.¹¹⁵

The problem with formalism (logicism and intuitionism inclusive) according to Majer Ulrich is that “they are biased and essentially incomplete”.¹¹⁶ One of the reasons for their being accused of biases is that one position (any of the three schools of mathematics) is formulated to be more agreeable and accepted than the other. For Majer, the dogma of the three schools of the philosophy of mathematics is “essentially incomplete because there was, and still exist besides Frege's Logicism, Brouwer's Intuitionism and Hilbert's Formalism at least one further position, namely Husserl's Phenomenological approach to the foundations of arithmetic which is also philosophically interesting”.¹¹⁷ Considering formalism within the context of the big three which Snapper calls the three crisis of mathematics, the question of where they lead us to or leaves us comes to mind. For Snapper, formalism and the other schools leaves us without a firm

^{113.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 148.

^{114.} Snapper, E “The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism”, 211

^{115.} Korner, S. *The Philosophy of Mathematics: An Introductory Essay*, 150.

^{116.} Majer, U. “Husserl: Between Frege's Logicism and Hilbert's Formalism” *The Basic International Yearbook of Cognition, Logic and Communication* (Vol. 4) 200 years of Analytic Philosophy, (August 2009), 1.

^{117.} Majer, U. “Husserl: Between Frege's Logicism and Hilbert's Formalism”, 1.

foundation for mathematics.¹¹⁸ In fact, when viewed from the perspective of “Gödel's incompleteness theorems and other results proved by Turing, Church, Skolem and Tarski in the 1930s, neither logicism nor formalism seemed a philosophically viable position. Intuitionism, whilst having its philosophical credentials intact, was unacceptable to most mathematicians because it involved discarding parts of classically accepted mathematics”.¹¹⁹ Snapper further stressed that “after Gödel's paper appeared in 1931, mathematicians on the whole threw up their hands in frustration and turned away from the philosophy of mathematics”.¹²⁰ The above notwithstanding, the influence exacted by these schools has remained strong since they have given us a new mathematics as well as become the basis of our collective philosophizing in the sub-field of the philosophy of mathematics.

Conclusion

In the preceding sections, attempts have been made to answer the question of what philosophy has to do with mathematics. It has accounted for the philosopher's interest in mathematics and shown the extent to which philosophy has helped in the task of grounding mathematics. Regardless of the attempt of philosophy, despite not having the capacity to tell the mathematician what to do and what not to do in the field of mathematics, the discipline of mathematics proceeds and thrives somewhat independently of the concerns of philosophers. While it is the task of the philosopher to hold the mathematician responsible for his axioms and findings, it is left for the mathematician in the final third to either make changes to his position or hold onto the same. However, there is a sense in which some mathematicians are concerned with philosophy and have perhaps employed it and allowed it to guide their activities in the enterprise of mathematics. In this sense,

^{118.} Snapper, E “The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism”, 216.

^{119.} Mary Tiles “*Philosophy of Mathematics*”, 345.

^{120.} Snapper, E “The Three Crisis in Mathematics: Logicism, Intuitionism and Formalism”, 216.

philosophy sets the direction of/for mathematical research. Drawing from the discourse and appraisal of the three basic schools that have been used to ground mathematics, it must be noted that despite the inability of any of the three schools—logicism, formalism, or intuitionism—to satisfactorily provide the foundation for mathematics, their contributions to the discourse and development of the subfield of the philosophy of mathematics cannot be gainsaid.

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